Ex :- Solve the following initial value problem

$(x-y)(dx+dy) = dx - dy \quad ; \quad y(0) = -1$

Sol:- We have $(x-y)(dx+dy) = dx - dy$

$\Rightarrow x\,dx - y\,dx - dx = -x\,dy + y\,dy - dy$

$\Rightarrow (x-y-1)\,dx = -(x-y+1)\,dy$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x-y-1}{-(x-y+1)}$ ——— (1)

Put $x - y = t \Rightarrow 1 - \dfrac{dy}{dx} = \dfrac{dt}{dx}$.

$\Rightarrow \boxed{\dfrac{dy}{dx} = 1 - \dfrac{dt}{dx}}$

(1) $\Rightarrow 1 - \dfrac{dt}{dx} = -\dfrac{t-1}{t+1} \Rightarrow \dfrac{dt}{dx} = 1 + \dfrac{t-1}{t+1}$

$\Rightarrow \dfrac{(t+1)\,dt}{2t} = dx$

$\Rightarrow \int \left(1 + \frac{1}{t}\right) dt = 2\int dx + C$

$\Rightarrow t + \log t = 2x + C \Rightarrow x - y + \log(x-y) = 2x + C$

$\Rightarrow \log(x-y) = x + y + C$ ——— (2)

$y(0) = -1$

(2) $\Rightarrow \log(0+1) = 0 - 1 + C \Rightarrow \boxed{C = 1}$

$\because \boxed{\log 1 = 0}$

$\therefore \log(x-y) = x + y + 1$

Homogeneous Differential Equations

A function $f(x,y)$ is said to be a homogeneous function in x and y of degree 'n' if

$\boxed{f(\lambda x, \lambda y) = \lambda^n f(x,y)}$ for every constant λ.

e.g:- $f(x,y) = \sin\frac{y}{x}$ is a homogeneous funᶜ of zero degree as.

$$f(\lambda x, \lambda y) = \sin\frac{\lambda y}{\lambda x} = \sin\frac{y}{x} = \lambda^0 f(x,y)$$

\# $f(x,y) = 3x^2 - 5xy + 7y^2$ is a homogeneous function of degree '2' As

$$f(\lambda x, \lambda y) = 3(\lambda x)^2 - 5(\lambda x)(\lambda y) + 7(\lambda y)^2$$

$$= \lambda^2(3x^2 - 5xy + 7y^2) = \lambda^2 f(x,y)$$

Defⁿ:- A First order differential eqⁿ is said to be homogeneous if it can be put in the form

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$; $f(x,y)$ & $g(x,y)$ are homogeneous funcˢ of same degree in x & y

OR//

A homogeneous function of the nth degree in x and y is that which can be put in the form $x^n f(y/x)$.

A differential eq^n of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

or $\frac{dx}{dx} = q\left(\frac{x}{y}\right)$. is homogeneous differential Eq^n.

To solve Homo. Differential eq^n:-

(i) put $y = vx$

; v is function of x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Separate the variables and integrate

Putting the value of $v = y/x$ in sol^n

In case $\frac{dx}{dy}$

put $x = vy$; v is function of y so that

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Separate the variables and integrate.

Putting the value of $v = x/y$ in sol^n.

Ex.:- 1 Solve :- $\frac{dy}{dx} = \frac{x+y}{x-y}$

Sol^n:- The given D.E. is : $\frac{dy}{dx} = \frac{x+y}{x-y}$ —— ①

given eq^n ① is homogeneous differential eq^n of order zero

$\because f(x,y) = \frac{x+y}{x-y}$

$\Rightarrow f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{\cancel{\lambda}(x+y)}{\cancel{\lambda}(x-y)}$

$\therefore f(\lambda x, \lambda y) = \lambda^0 f(x,y)$

now put $y = vx$ —— ②

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$ —— ③

From ①, ② & ③, we get

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx} = \frac{\cancel{x}(1+v)}{\cancel{x}(1-v)}$$

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$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1+v}{1-v} - v = \dfrac{1+v-v(1-v)}{1-v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1+\cancel{v}-\cancel{v}+v^2}{1-v} = \dfrac{1+v^2}{1-v}$

$\Rightarrow \left(\dfrac{1-v}{1+v^2}\right)dv = \dfrac{dx}{x}$

$\Rightarrow \int\left(\dfrac{1}{1+v^2} - \dfrac{2v}{2(1+v^2)}\right)dv = \int\dfrac{dx}{x} + C$

$\Rightarrow \tan^{-1}v - \frac{1}{2}\log|1+v^2| = \log x + C$

$\Rightarrow \tan^{-1}\dfrac{y}{x} - \dfrac{1}{2}\log\left|1+\dfrac{y^2}{x^2}\right| = \log x + C$

$\Rightarrow \tan^{-1}\dfrac{y}{x} - \dfrac{1}{2}\left\{\log(x^2+y^2) - \log x^2\right\} = \log x + C$

$\Rightarrow \tan^{-1}y/x - \frac{1}{2}\log(x^2+y^2) + \cancel{\log(x^2)^{1/2}} = \cancel{\log x} + C$

$\Rightarrow \tan^{-1}y/x - \frac{1}{2}\log(x^2+y^2) = C$

which is the required sol^n

Ex.:- Solve: $x^2y\,dx - (x^3+y^3)\,dy = 0$

sol^n:- The given D.E. is

$x^2y\,dx - (x^3+y^3)\,dy = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x^2y}{x^3+y^3}$ —— (*)

Given D.E. is homogeneous eq. of order zero

put $y = vx$ —— ①

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ —— ②

(A) $\Rightarrow v + x\frac{dv}{dx} = \frac{x^2(vx)}{x^3+(vx)^3} = \frac{x^3 v}{x^3(1+v^3)}$

$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{\cancel{v}-\cancel{v}-v^4}{1+v^3}$

$\Rightarrow \left(\frac{1+v^3}{v^4}\right)dv = -\frac{dx}{x} \Rightarrow \int\left(\frac{1}{v^4}+\frac{v^3}{v^4}\right)dv = -\int\frac{dx}{x} + C$

$\Rightarrow \frac{v^{-3}}{-3} + \log|v| = -\log x + C$

$\Rightarrow -\frac{1}{3v^3} + \log v + \log x = C$

$\Rightarrow -\frac{1}{3}\left(\frac{x^3}{y^3}\right) + \log\left(\frac{y}{x}\right) + \log x = C$

$\Rightarrow -\frac{1}{3}\left(\frac{x^3}{y^3}\right) + \log(y) - \cancel{\log(x)} + \cancel{\log(x)} = c$

$\Rightarrow \frac{-x^3}{3y^3} + \log(y) = C$ which is the required solⁿ.

Ex:- solve $(1+e^{\frac{x}{y}})dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

solⁿ:- The given D.E. $(1+e^{\frac{x}{y}})dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

$\Rightarrow e^{x/y}\left(\frac{x}{y} - 1\right)dy = \left(1+e^{x/y}\right)dx.$

$\Rightarrow \frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{(1+e^{x/y})}$ —— (A)

Given eq^n is the of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$ & hence homo.

Put $x = vy$ — ①

$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$ — ②

(A) $\Rightarrow v + y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}$

$\Rightarrow y\frac{dv}{dy} = \frac{e^v v - e^v - v(1+e^v)}{1+e^v} = \frac{\cancel{e^v v} - e^v - v - \cancel{e^v v}}{1+e^v}$

$\Rightarrow y\frac{dv}{dy} = \frac{-(e^v + v)}{1+e^v} \Rightarrow \left(\frac{1+e^v}{e^v+v}\right)dv = -\frac{dy}{y}$

$\Rightarrow \int \frac{1+e^v}{v+e^v}\,dv = -\int\frac{dy}{y} + c$

$\Rightarrow \log(v+e^v) = -\log y + \log c$

$\Rightarrow \log(v+e^v) = \log\left(\frac{c}{y}\right)$

$\Rightarrow v + e^v = \frac{c}{y} \Rightarrow \left(\frac{x}{y} + e^{x/y}\right) = \frac{c}{y}$

$\Rightarrow \frac{(x + ye^{x/y})}{\cancel{y}} = \frac{c}{\cancel{y}}$

$\therefore x + ye^{x/y} = c$ which is the required sol^n.